MVT, Optimization, L'Hopital's rule and Integrals

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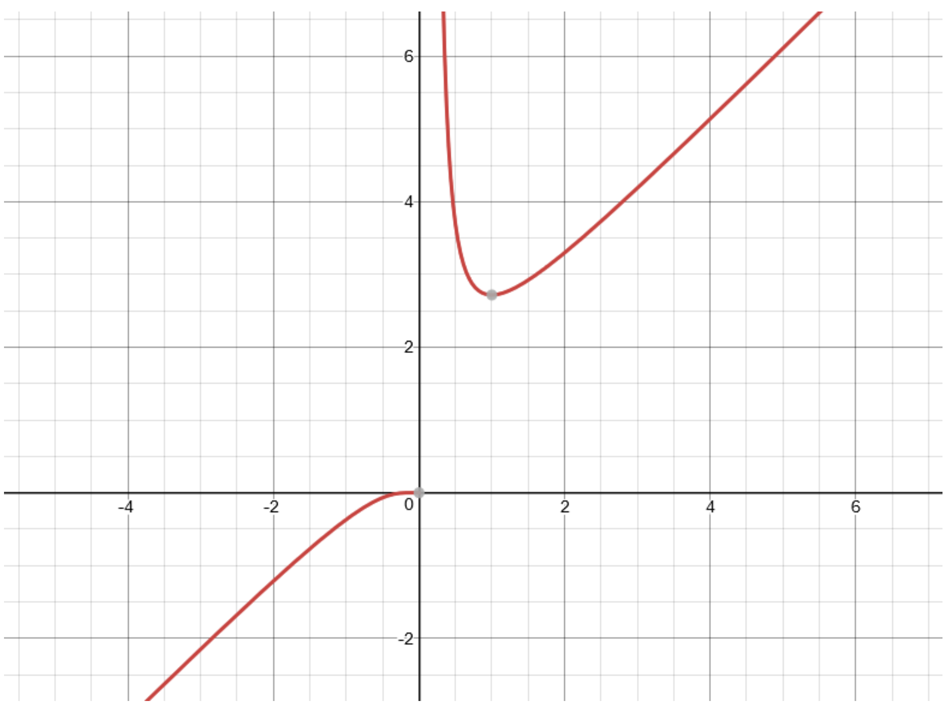
Problems

Problem 1. Suppose $0 < f'(x) < \frac{1}{2}$ for all x-values. Show that $f(-1) < f(1) < 1 + f(-1)$.

Solution: By Mean Value Theorem, $\frac{f(1)-f(-1)}{1-(-1)} = f'(c)$ for some $c \in (-1, 1)$. Since $f'(c) > 0$, we get $\frac{f(1)-f(-1)}{2} > 0$, i.e. $f(1) > f(-1)$. Since $f'(c) < \frac{1}{2}$, we get $\frac{f(1)-f(-1)}{2} < \frac{1}{2}$, i.e. $f(1) < 1 + f(-1)$.

Problem 2. Sketch the graph of $xe^{1/x}$.

Solution: The domain is the set $\{x \in \mathbb{R} \mid x \neq 0\}$ of all non-zero real numbers.
To find the critical points, we set $f'(x) = e^{1/x} - \frac{1}{x}e^{1/x} = \frac{1}{x}e^{1/x}(x-1) = 0$, which has only one solution $x = 1$. Since at $x = 1$ the function $f'(x)$ changes sign from $-$ to $+$, the point $x = 1$ is a local minimum.
To find inflection points, we find $f''(x) = \frac{1}{x^3}e^{1/x}$ which is never zero. At $x = 0$ the derivative doesn't exist (even the function is not defined), but $f''(x)$ changes sign from $-$ to $+$ at this point, and so $x = 0$ is an inflection point. The function is concave down when $x < 0$ and concave up when $x > 0$.
Now, we need to analyze the only point of discontinuity: the point $x = 0$. We find $\lim\limits_{x \to 0^+} xe^{1/x} = +\infty$ and $\lim\limits_{x \to 0^-} xe^{1/x} = 0$.
Finally, we need to understand the behavior of $xe^{1/x}$ when $x \to +\infty$ and when $x \to -\infty$. We see $\lim\limits_{x \to +\infty} xe^{1/x} = +\infty$ and $\lim\limits_{x \to -\infty} xe^{1/x} = -\infty$.
Combining it all together, we get the following graph:



Problem 3. An open-topped cylindrical pot is to have volume 250 cm^3.The material for the bottom of the pot costs 4 cents per cm^2; that for its curved side costs 2 cents per cm^2. What dimensions will minimize the total cost of this pot?

Solution: Let r be the radius of the pot, and h be its height. Then the volume is given by $V = \pi r^2 h$, and so we are given that $\pi r^2 h = 250$. From this condition we can find that $h = \frac{250}{\pi r^2}$.
The area of the bottom of the pot is πr^2 and so it will cost $4\pi r^2$ to make it. The area of the side of the pot is $2\pi r h = \frac{500}{r}$ and so it will cost $\frac{1000}{r}$ to make it. The total cost is $C(r) = 4\pi r^2 + \frac{1000}{r}$. We want to minimize this cost.
The derivative $C'(r) = 8\pi r - \frac{1000}{r^2}$, and so the critical points are found by equation $8\pi r - \frac{1000}{r^2} = 0$, which has only one solution $r = \frac{5}{\sqrt[3]{\pi}}$. Since the derivative changes sign at $r = \frac{5}{\sqrt[3]{\pi}}$ from $-$ to $+$, this is a point of local minimum. Since $\lim_{r\to 0^+} C(r) = \infty$ and $\lim_{r\to +\infty} C(r) = \infty$, the point $r = \frac{5}{\sqrt[3]{\pi}}$ is also a point of **global** minimum, and so the optimal radius should be $r = \frac{5}{\sqrt[3]{\pi}}$. The corresponding height is
$h = \frac{250}{\pi r^2} =$ whatever it is .

Problem 4. Compute $\lim_{x\to 0} \left(\frac{1}{\sin^2(x)} - \frac{1}{x^2}\right)$.

Solution: We get

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin^2(x)} - \frac{1}{x^2}\right) &= \lim_{x\to 0} \frac{x^2 - \sin^2(x)}{x^2 \sin^2(x)} \\
&= \lim_{x\to 0} \frac{x^2 - \sin^2(x)}{x^2 \sin^2(x)} \cdot \lim_{x\to 0} \frac{\sin^2(x)}{x^2} \\
&= \lim_{x\to 0} \frac{x^2 - \sin^2(x)}{x^4} \\
&= \lim_{x\to 0} \frac{2x - 2\sin(x)\cos(x)}{4x^3} \\
&= \lim_{x\to 0} \frac{2x - \sin(2x)}{4x^3} \\
&= \lim_{x\to 0} \frac{2 - 2\cos(2x)}{12x^2} \\
&= \lim_{x\to 0} \frac{4\sin(2x)}{12\cdot 2x} \\
&= \frac{1}{3}\lim_{x\to 0} \frac{\sin(2x)}{2x} \\
&= \frac{1}{3}.
\end{aligned}$$

Problem 5. Compute the following integrals:

1. $\int x e^{x^2} dx$
2. $\int \frac{x+1}{2x-3} dx$
3. $\int \frac{1}{x^2+4} dx$

Solution:

1. $\frac{1}{2} e^{x^2} + C$.
2. since $\frac{x+1}{2x-3} = \frac{1}{2} + \frac{5}{4x-6}$, we can guess the integral to be $\frac{1}{2}x + \frac{5}{4}\ln|4x-6| + C$.
3. we know the derivative of $\arctan(x)$ is $\frac{1}{x^2+1}$, so if we take $\arctan(x/2)$, its derivative will be $\frac{1}{2}\cdot\frac{1}{(x/2)^2+1}$ which is almost what we want, except we need to multiply it by $\frac{1}{2}$. Thus, the answer is $\frac{1}{2}\arctan\frac{x}{2} + C$.

Problem 6. At time $t = 0$ a car is moving at 6 m/s and driver smoothly accelerates so that the acceleration after t seconds is $a(t) = 3t$ m/s^2.

1. Write a formula for the speed $v(t)$ of the car after t seconds.
2. How far did the car travel between during the time it took to accelerate from 6 m/s to 30 m/s?

Solution:

1. Since $a(t) = \frac{dv(t)}{dt} = 3t$, we know $v(t) = \frac{3}{2}t^2 + C$. Since at $t = 0$ the speed is 6 m/s, $C = 6$, and so $v(t) = \frac{3}{2}t^2 + 6$.
2. The velocity was 6 m/s at time $t = 0$ and it was 30 m/s when $\frac{3}{2}t^2 + 6 = 30$, i.e. $t = 4$. So, we are interested in finding $s(4) - s(0)$. Since $v(t) = \frac{ds(t)}{dt}$, we find $s(t) = \frac{1}{2}t^3 + 6t + C$, and $s(4) - s(0) = \frac{1}{2}4^3 + 6 \cdot 4 + C - \frac{1}{2}0^3 - 6 \cdot 0 - C = 56$ meters.